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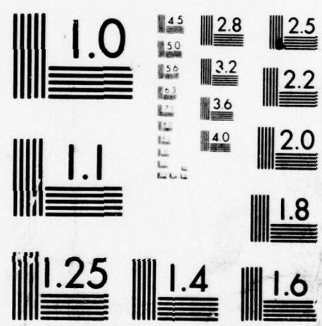

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Multistate Reliability Models,

by

(10) William S. Griffith Henry W. Black

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Multistate Reliability Models[1]

by

William S. Griffith

Abstract

In this paper an axiomatic development of multistate systems is presented. Three types of coherence based on the strength of the relevancy axiom are studied. The strongest of these has been investigated previously by El-Neweihi, Proschan, and Sethuraman. One of the weaker types of coherence permits wider applicability to real life situations without sacrificing any of the mathematical results obtained by El-Neweihi, Proschan, and Sethuraman. The concept of system performance is formalized through expected utility and the effect of component improvement on system performance is studied using a generalization of Birnbaum's reliability importance.

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1. Introduction

Most of the reliability literature deals with binary systems of binary components in which the only two states are functioning and failed. Some recent work by Barlow and Wu [2], El-Neweihi, Proschan, and Sethuraman [3] (referred to as EPS throughout), and Ross [4] treat the general case of more than two states. This idea is quite useful since in many real life situations components and/or systems can be in intermediate states. In addition, multistate systems can be used to incorporate the idea of spare parts where a component's being in state j means that in addition to the presently operating component, there are $j-1$ spares.

Ross considers components which assume arbitrary real values and investigates stochastic and dynamic aspects of the system. Barlow's definition of multistate systems depends on a set theoretic decomposition which arises in the binary case. EPS adopt an axiomatic definition which generalizes the axiomatic definition of a binary coherent system. The multistate coherent systems in their paper include as a subclass those of Barlow and Wu. However the relevancy axiom of EPS is very strong. In this paper, we consider this axiom and two weaker relevancy axioms which allow a certain real life model which is not permitted under the strongest axiom. Furthermore all of the results of EPS hold under one of the weaker assumptions. Deterministic models are developed in section 2. In section 3 the stochastic performance of a system is considered in terms of expected utility while in section 4, a generalization of Birnbaum's reliability importance is investigated.

The notation of Barlow and Proschan [1] and EPS [3] is used.

2. Deterministic Properties of Multistate Monotone Systems

We begin by making a formal definition of a multistate monotone system which generalizes the notion of a monotone system in the binary case.

Definition 2.1. Let ϕ be a function with domain $\{0,1,\ldots, M\}^n$ and range $\{0,1,\ldots, M\}$ where M and n are positive integers. Then ϕ is said to be a multistate monotone system (MMS) if it satisfies

(i) $\phi(\underline{x})$ is increasing in $\underline{x} \geq \underline{0}$

(ii) $\min\limits_{1\leq i\leq n} x_i \leq \phi(\underline{x}) \leq \max\limits_{1\leq i\leq n} x_i$

It is easy to see that (i) implies that $\phi(\underline{x}\vee\underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y})$ for all $\underline{x},\underline{y} \geq 0$. Furthermore (i) is implied by this condition since if $\underline{y} \geq \underline{x}$, then $\phi(\underline{y}) = \phi(\underline{x}\vee\underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y}) \geq \phi(\underline{x})$. Analogous remarks show that the inequality $\phi(\underline{x}\wedge\underline{y}) \leq \phi(\underline{x}) \wedge \phi(\underline{y})$ for all $\underline{x},\underline{y} \geq \underline{0}$ is also equivalent to (i).

Under the assumption of (i) it is easy to verify that (ii) is equivalent to the condition that $\phi(\underline{k}) = k$ for all $k \in \{0,1,\ldots, M\}$.

We next consider three relevance axioms. The strongest of these has been considered by EPS. This leads to a type of coherence which we shall call strong coherence. The successively weaker types of coherence will be called coherence and weak coherence.

Definition 2.2. Let $\phi(\underline{x})$ be a MMS. If:

(i) for any component i and state j, there exists $\underline{x}$ such that $\phi(j_i,\underline{x}) = j$ while $\phi(\ell_i,\underline{x}) \neq j$ for $\ell \neq j$, then $\phi(\underline{x})$ is said to be strongly coherent.

(ii) for any component i and state $j \geq 1$, there exists $\underline{x}$ such that $\phi((j-1)_i, \underline{x}) < \phi(j_i, \underline{x})$, then $\phi(\underline{x})$ is said to be coherent.

(iii) for any component i and state j, there exists $\underline{x}$ such that $\phi(j_i, \underline{x}) \neq \phi(\ell_i, \underline{x})$ for some $\ell \neq j$, then $\phi(\underline{x})$ is said to be weakly coherent.

By taking $j = 0$ in condition (iii) we see that for any component i, there exists $\underline{x}$ such that $\phi(0_i, \underline{x}) < \phi(\ell_i, \underline{x})$ for some $\ell \neq 0$ and consequently $\phi(0_i, \underline{x}) < \phi(M_i, \underline{x})$. Conversely if for any component i, there exists $\underline{x}$ such that $\phi(0_i, \underline{x}) < \phi(M_i, \underline{x})$, then for any state j, $0 < j < M$, $\phi(j_i, \underline{x})$ cannot be equal to both $\phi(0_i, \underline{x})$ and $\phi(M_i, \underline{x})$. Therefore we see that condition (iii) may be replaced by the equivalent condition:

(iii') for any component i, there exists $\underline{x}$ such that $\phi(0_i, \underline{x}) < \phi(M_i, \underline{x})$.

In practice condition (iii') may be easier to check.

It is obvious that (i) implies (ii) which in turn implies (iii) in Definition 2.2. We next present examples which show that neither of the reverse implications holds.

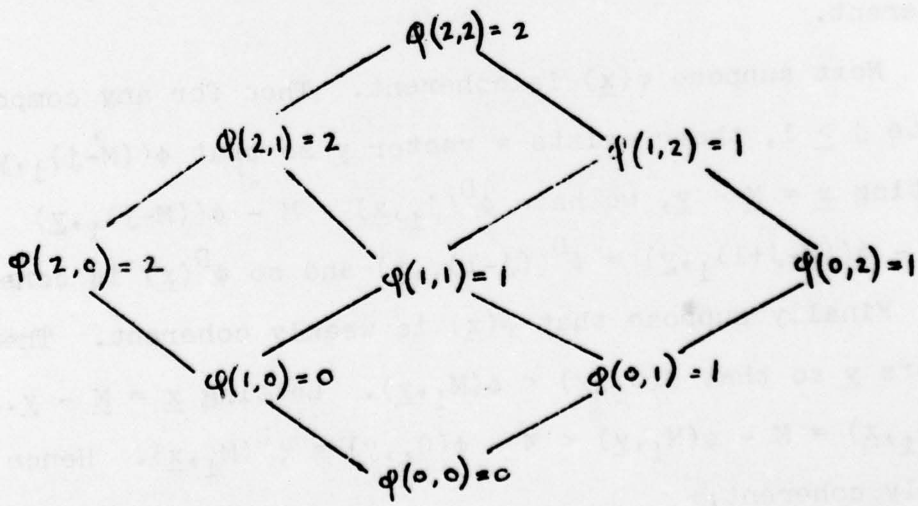
Example 2.1. A MMS which is coherent but not strongly coherent.

Consider a system of $n(>1)$ components with $M + 1(>2)$ states having structure function $\phi(\underline{x}) = \left[\frac{1}{n} \sum_{i=1}^{n} x_i\right]$ where $[\cdot]$ is the greatest integer function. It is easily seen to be a monotone system. To see that it is coherent, note that $\phi(j_i, \underline{j}) = j$ while $\phi((j-1)_i, \underline{j}) = j - 1$. Next we show that it cannot be strongly coherent. To be strongly coherent, for any component i and state $j \in \{1, \ldots, M-1\}$, there would have to exist $\underline{x}$ such that $\phi((j-1)_i,\underline{x}) \leq j - 1 < j = \phi(j_i,\underline{x})$ $= j < j + 1 \leq \phi((j+1)_i,\underline{x})$. That is, $\left[\frac{j-1}{n}+\frac{1}{n}\sum_{k\neq i} x_k\right] \leq j - 1 < j$ $= \left[\frac{j}{n}+\frac{1}{n}\sum_{k\neq i} x_k\right] = j < j + 1 \leq \left[\frac{j+1}{n}+\frac{1}{n}\sum_{k\neq i} x_k\right]$. Thus, $\frac{j-1}{n} + \frac{1}{n} \sum_{k\neq i} x_k < j < j + 1 \leq \frac{j+1}{n} + \frac{1}{n} \sum_{k\neq i} x_k$ and $\frac{j+1}{n} + \frac{1}{n} \sum_{k\neq i} x_k$ $- (\frac{j-1}{n}+\frac{1}{n}\sum_{k\neq i} x_k) = \frac{2}{n} > 1$. This cannot happen since $n \geq 2$.

The structure function of this example can be used to stochastically model the following physical situation. Consider a power generating system consisting of n power plants. Each plant is either operational or not operational and the system state is taken to be the number of operational plants (and thus M=n). The components assume the values 1,..., M − 1 with zero probability. Thus it seems desirable to have a definition of coherence which is general enough to include this structure function.

Example 2.2. A weakly coherent system which is not coherent.

Consider $\phi(\underline{x})$ defined by the lattice below.



Clearly $\phi(\underline{x})$ is monotone. Further, $\phi(0,0) < \phi(2,0)$ and $\phi(0,0) < \phi(0,2)$ so that $\phi(\underline{x})$ is weakly coherent. But $\phi(1,x_2) = \phi(0,x_2)$ for $x_2 = 0,1$ and 2. Consequently, $\phi(\underline{x})$ is not coherent.

As with binary systems, a concept of duality may be defined.

<u>Definition 2.3</u>. For any MMS $\phi(\underline{x})$ we define the <u>dual</u> of $\phi(\underline{x})$ by the equation $\phi^D(\underline{x}) = M - \phi(\underline{M}-\underline{x})$.

It is easy to verify that the dual of a MMS is also a MMS. Furthermore we show that the dual inherits the type of coherence possessed by the original MMS.

<u>Proposition 2.1</u>. For any MMS $\phi(\underline{x})$, the dual $\phi^D(\underline{x})$ possesses the same type of coherence as $\phi(\underline{x})$.

<u>Proof</u>. First, suppose that $\phi(\underline{x})$ is strongly coherent. Then for any component i and state M - j, there exists $\underline{y}$ such that $\phi((M-j)_i,\underline{y}) = M - j$ while $\phi((M-\ell)_i,\underline{y}) \neq M - j$ for all $\ell \neq j$. Letting $\underline{x} = \underline{M} - \underline{y}$, we have $\phi^D(j_i,\underline{x}) = M - \phi((M-j)_i,\underline{y}) = j$ while $\phi^D(\ell_i,\underline{x}) = M - \phi((M-\ell)_i,\underline{y}) \neq j$ for $\ell \neq j$. Thus $\phi^D(\underline{x})$ is strongly coherent.

Next suppose $\phi(\underline{x})$ is coherent. Then for any component i and state $j \geq 1$, there exists a vector $\underline{y}$ so that $\phi((M-j)_i,\underline{y}) < \phi((M-j+1)_i,\underline{y})$. Letting $\underline{x} = \underline{M} - \underline{y}$, we have $\phi^D(j_i,\underline{x}) = M - \phi((M-j)_i,\underline{y})$ $> M - \phi((M-j+1)_i,\underline{y}) = \phi^D((j-1)_i,\underline{x})$ and so $\phi^D(\underline{x})$ is coherent.

Finally suppose that $\phi(\underline{x})$ is weakly coherent. Then there exists $\underline{y}$ so that $\phi(0_i,\underline{y}) < \phi(M_i,\underline{y})$. Letting $\underline{x} = \underline{M} - \underline{y}$, we have $\phi^D(0_i,\underline{x}) = M - \phi(M_i,\underline{y}) < M - \phi(0_i,\underline{y}) = \phi^D(M_i,\underline{x})$. Hence $\phi^D(\underline{x})$ is weakly coherent.

In the next definition we generalize the concept of a path vector.

Definition 2.4. Let $\phi(\underline{x})$ be a MMS. If $\phi(\underline{x}) = j$, the $\underline{x}$ is called a state j vector. If in addition, $\phi(\underline{y}) < j$ for all $\underline{y} < \underline{x}$, then $\underline{x}$ is called a minimal state j vector.

The existence of such minimal state j vectors follows from the definition. Further, it is clear that $\phi(\underline{y}) \geq j$ if and only if $\underline{y} \geq \underline{x}$ for some minimal state j vector $\underline{x}$.

By monotonicity, it follows that $\phi(\underline{x} \vee \underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y})$ and $\phi(\underline{x} \wedge \underline{y}) \leq \phi(\underline{x}) \wedge \phi(\underline{y})$. In the next proposition, we investigate the implications of equality.

Proposition 2.2. Let $\phi(\underline{x})$ be a coherent system. Then:

(i) $\phi(\underline{x} \vee \underline{y}) = \phi(\underline{x}) \vee \phi(\underline{y})$ for all $\underline{x}, \underline{y} \geq 0$ if and only if $\phi(\underline{x}) = \max_{1 \leq i \leq n} x_i$.

(ii) $\phi(\underline{x} \wedge \underline{y}) = \phi(\underline{x}) \wedge \phi(\underline{y})$ for all $\underline{x}, \underline{y} \geq 0$ if and only if $\phi(\underline{x}) = \min_{1 \leq i \leq n} x_i$.

Proof: In both cases the "if" part is easy. To demonstrate the "only if" part of (i) we assume that $\phi(\underline{x} \vee \underline{y}) = \phi(\underline{x}) \vee \phi(\underline{y})$ for all $\underline{x}, \underline{y} \geq \underline{0}$. By coherence we have that for any component i and state $j \geq 1$, there exists a vector $\underline{x}$ such that $\phi((j-1)_i, \underline{x}) < \phi(j_i, \underline{x})$. But $\phi(j_i, \underline{x}) = \max\{\phi(j_i, \underline{0}), \max_{k \neq i} \phi((x_k)_k, \underline{0})\}$ and $\phi((j-1)_i, \underline{x})$ $= \max\{\phi(j-1)_i, \underline{0}), \max_{k \neq i} \phi((x_k)_k, \underline{0}))\}$. Hence $\phi((j-1)_i, \underline{0}) < \phi(j_i, \underline{0})$. This holds for all states $j \geq 1$. Hence $0 = \phi(\underline{0}) < \phi(1_i, \underline{0})$

$<\ldots< \phi(j_1,\underline{0}) \leq j$. This string of inequalities forces $\phi(j_1,\underline{0}) = j$ for any component i. Then $\phi(\underline{x}) = \max_{1\leq i\leq n} \{\phi((x_i)_i,\underline{0})\} = \max_{1\leq i\leq n} x_i$.

The "only if" part of (ii) follows from (i) and the concepts of duality.

We note that the conclusions of Proposition 2.2 hold if $\phi(\underline{x})$ is strongly coherent since strongly coherent implies coherent. The next example will show that the hypothesis of Proposition 2.2 cannot be weakened to weakly coherent systems.

Example 2.3. Consider once again the system of example 2.2. It was shown to be weakly coherent but not coherent. Further, since $\phi(0,2) = 1$, $\phi(\underline{x})$ is not $\max_{1\leq i\leq n} x_i$. We will now show that it satisfies $\phi(\underline{x}\vee\underline{y}) = \phi(\underline{x}) \vee \phi(\underline{y})$ for all $\underline{x},\underline{y} \geq \underline{0}$. By its monotonicity, $\phi(\underline{x}\vee\underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y})$. To complete the proof it suffices to show two implications. One is that if $\phi(\underline{x}\vee\underline{y}) = 2$ then $\phi(\underline{x}) \vee \phi(\underline{y}) = 2$. The second is that if $\phi(\underline{x}) \vee \phi(\underline{y}) = 0$, then $\phi(\underline{x}\vee\underline{y}) = 0$. First assume that $\phi(\underline{x}\vee\underline{y}) = 2$. Then $\underline{x} \vee \underline{y} \geq (2,0)$ and thus $x_1 = y_1 = 2$. Thus $\phi(\underline{x}) = \phi(\underline{y}) = 2$ and $\phi(\underline{x}) \vee \phi(\underline{y}) = 2$. Now assume $\phi(\underline{x}) \vee \phi(\underline{y}) = 0$. Then $\phi(\underline{x}) = \phi(\underline{y}) = 0$. Hence $\underline{x},\underline{y} \leq (1,0)$, so that $x_2 = y_2 = 0$ and $x_1 \vee y_1 \leq 1$. Therefore $\phi(\underline{x}\vee\underline{y}) = 0$.

Next we consider the idea of modules in the multistate setting. We are interested here in determining whether the relevancy of a component within a module and the relevancy of that module within a larger system would entail the relevancy of the component within the larger system. We first show by an example that this will not necessarily occur in the case of weak relevancy.

<u>Example 2.4</u>. Consider again the structure function of example 2.2. If we define $\psi(x_1,x_2,x_3) = \phi(\phi(x_1,x_2),x_3)$, it can be shown that $\psi(x_1,0,x_3) = \psi(x_1,2,x_3)$ for all x_1 and x_3.

For coherent and strongly coherent systems it is easy to show that a relevant component within a relevant module is relevant in the system. For coherent systems this follows from the fact that if component i of a module $\psi(\underline{x})$ is relevant, then for all states $j \geq 1$, there exists a vector $\underline{x}$ such that $\psi((j-1)_i,\underline{x}) < \psi(j_i,\underline{x})$. Further there exists a vector $\underline{y}$ such that $\phi(\psi(j_i,\underline{x})-1,\underline{y}) < \phi(\psi(j_i,\underline{x}),\underline{y})$. Thus $\phi(\psi((j-1)_i,\underline{x}),\underline{y}) < \phi(\psi(j_i,\underline{x}),\underline{y})$ and component i of the module is relevant in the system.

For strongly coherent systems, if component i of a module $\psi(\underline{x})$ satisfies the strong relevancy condition then for any state j, there exists a vector $\underline{x}$ with $\psi(j_i,\underline{x}) = j$ while $\psi(\ell_i,\underline{x}) \neq j$ for $\ell \neq j$. Next there exists a vector $\underline{y}$ such that $\phi(\psi(j_i,\underline{x}),\underline{y}) = j$ and $\phi(k_i,\underline{y}) \neq j$ for $k \neq \psi(j_i,\underline{x})$. That is, $\phi(\psi(\ell_i,\underline{x}),\underline{y}) \neq j$ for $\ell \neq j$. Thus, component i of the module satisfies the strong relevancy condition within the larger system.

3. Stochastic Performance of a System

In this section we consider the state of component i, represented by X_i to be a random variable. Further we assume that for distinct i and j that X_i and X_j are independent. We are interested in making precise the way in which improved component performance affects system performance. There are two ways to parameterize component performance. One is to let $p_{ij} = P\left[X_i = j\right]$ for $j = 1, 2, \ldots, M$, ($p_{i0} = 1 - \sum_{j=1}^{M} p_{ij}$). Then the vector $(p_{i1}, \ldots, p_{iM})$ describes the distribution of X_i. Another parameterization is to let $\rho_{ij} = \sum_{\ell=j}^{M} p_{i\ell}$ for $j \geq 1$. Then $1 \geq \rho_{i1} \geq \ldots \geq \rho_{iM} \geq 0$. Given $(p_{i1}, \ldots, p_{iM})$ one can determine $(\rho_{i1}, \ldots, \rho_{iM})$ and conversely. We shall choose to use the ρ_{ij}'s as parameter since improving component i (i.e. the improved component is stochastically better than the original component) is equivalent to increasing all of the ρ_{ij}'s. We use the concept of expected utility to describe system performance since this results in wider applicability than the use of the expected value.

Now let $h_j(\underline{\rho}) = P\left[\phi(\underline{X}) = j\right]$ for $j \geq 0$ and let $0 \leq a_0 \leq a_1 \leq \ldots \leq a_M$ represent the utilities attached to the various states of the system. We shall assume $a_0 = 0$ without loss of generality or applicability. Then the expected utility $U(\underline{\rho})$ is $U(\underline{\rho}) = \sum_{j=1}^{M} a_j h_j(\underline{\rho})$. If $a_j = j$ for all j, then $U(\underline{\rho}) = E_{\underline{\rho}}\left[\phi(\underline{X})\right]$. We note at this time that if $b_1 = a_1$ and $b_k = a_k - a_{k-1}$ for $k = 2, \ldots, M$ then

$$U(\underline{\rho}) = \sum_{j=1}^{M} a_j h_j(\underline{\rho}) = \sum_{i=1}^{M} b_i H_i(\underline{\rho}) \text{ where } H_i(\underline{\rho}) = \sum_{j=i}^{M} h_j(\underline{\rho}) = P\left[\phi(x) \geq i\right].$$

We first mathematically formalize the idea that improved component performance results in improved system performance.

<u>Lemma 3.1</u>. Let $\phi(\underline{x})$ be a MMS and $\underline{\rho}$ and $\underline{\rho}^*$ be two distributions with $\rho_{kj} = \rho^*_{kj}$ for all j and for all $k \neq i$ and $\rho_{ij} \leq \rho^*_{ij}$ for all j. Then for any $a \in \{0,1,\ldots, M\}$ $P[\phi(\underline{X}) \geq a] \leq P^*[\phi(\underline{X}) \geq a]$ where P is computed under $\underline{\rho}$ and P* is computed under $\underline{\rho}^*$.

<u>Proof</u>:

Consider independent random variables $X_1, \ldots, X_n$ $(X^*_1, \ldots, X^*_n)$ with distribution given by $\underline{\rho}$(by $\underline{\rho}^*$). Then $X_i \overset{st}{\leq} X^*_i$ and $X_k \overset{st}{=} X^*_k$ for $k \neq i$. Since ϕ is an increasing function in each of its components, $\phi(X) \overset{st}{\leq} \phi(X^*)$. Then for any $a \in \{0,1,\ldots, M\}$, $P[\phi(\underline{X}) \geq a] \leq P[\phi(\underline{X}^*) \geq a] = P^*[\phi(\underline{X}) \geq a]$.

<u>Lemma 3.2</u>. Let $\phi(\underline{x})$ be a monotone system and $\underline{\rho}$ and $\underline{\rho}^*$ be two probability distributions such that $\rho_{i\ell} \leq \rho^*_{i\ell}$ for all i and $\ell \geq 1$. Then $P[\phi(\underline{X}) \geq a] \leq P^*[\phi(\underline{X}) \geq a]$ for all $a \in \{0,1,\ldots, M\}$.

<u>Proof</u>: Omitted.

From these lemmas we have a proposition formalizing the notion of system improvement in terms of expected utilities.

<u>Proposition 3.1</u>. Consider a monotone system $\phi(\underline{x})$ with associated utilities $0 = a_0 \leq a_1 \leq \cdots \leq a_M$. Suppose that $\underline{\rho}$ and $\underline{\rho}^*$ are two probability distributions with $\rho_{i\ell} \leq \rho^*_{i\ell}$ for all $\ell \geq 1$ and all i. Then $U(\underline{\rho}) \leq U(\underline{\rho}^*)$.

<u>Proof</u>: Omitted

4. Reliability Importance in Multistate Systems

A basic concept of component importance is due to Birnbaum (see Chapter 2 of [1]). In this section we attempt to generalize Birnbaum's reliability importance to the multistate setting. We begin by recalling some of the properties of Birnbaum's reliability importance for binary systems of binary components. If $h(\underline{p})$ is the system reliability function, then:

(i) $I(i) = \frac{\partial h}{\partial p_i}$ (where this partial derivative is evaluated at the present reliabilities).

(ii) $I(i) = h(1_i,\underline{p}) - h(0_i,\underline{p}) = E\left[\phi(1_i,\underline{X})\right] - E\left[\phi(0_i,\underline{X})\right]$.

(iii) $I(i) = P\left[\phi(1_i,\underline{X})=1 \text{ and } \phi(0_i,\underline{X})=0\right]$.

(iv) $h(\underline{p}) = p_i I(i) + h(0_i,\underline{p})$ (a version of the pivotal decomposition).

Notice that (iv) implies that $h((p_i+\Delta)_i,\underline{p}) = h(\underline{p}) + \Delta I(i)$ thus demonstrating that a component improvement of Δ in component i yields a system improvement of $\Delta I(i)$. We shall see that there is a vector which generalizes these properties to the multistate setting.

Consider a multistate system with expected utility $U(\underline{\rho}) = \sum_{j=1}^{M} a_j h_j(\underline{\rho}) = \sum_{j=1}^{M} b_j H_j(\underline{\rho})$ as before where $b_j \geq 0$ for all j. Define $I_{\ell j}(i) = P\left[\phi(\ell_i,\underline{X})\geq j\right] - P\left[\phi((\ell-1)_i,\underline{X})\geq j)\right]$ for all i and all ℓ and $j \geq 1$. Further define $I_\ell(i) = \sum_{j=1}^{M} b_j I_{\ell j}(i)$ and $\underline{I}(i) = (I_1(i),I_2(i),\ldots, I_M(i))$. We shall call $\underline{I}(i)$ the importance vector of component i.

<u>Proposition 4.1</u> (General Decomposition).

$U(\underline{\rho}) = \sum_{j=1}^{M} b_j P\left[\phi(0_i,\underline{X})\geq j\right] + \underline{I}(i)\cdot\underline{\rho}_i^T$ where $\underline{\rho}_i = (\rho_{i1},\rho_{i2},\ldots,\rho_{iM})$.

<u>Proof</u>: We write $P\left[\phi(\underline{X})\geq j\right]$ as $\sum_{\ell=0}^{M} P\left[\phi(\ell_i,\underline{X})\geq j\right] P\left[X_i=\ell\right]$ and replace $P\left[X_i=\ell\right]$ by $\rho_{i\ell} - \rho_{i,\ell+1}$. By a rearrangement of terms and a change of variables it can be shown that $P\left[\phi(\underline{X})\geq j\right] = P\left[\phi(0_i,\underline{X})\geq j\right]$ $+ \sum_{\ell=1}^{M}\{P\left[\phi(\ell_i,\underline{X})\geq j\right]-P\left[\phi((\ell-1)_i,\underline{X})\geq j\right]\}\rho_{i\ell}$. Then $U(\underline{\rho}) = \sum_{j=1}^{M} b_j P\left[\phi(0_i,\underline{X})\geq j\right]$ $+ \sum_{\ell=1}^{M}\sum_{j=1}^{M} b_j\{P\left[\phi(\ell_i,\underline{X})\geq j\right]-P\left[\phi((\ell-1)_i,\underline{X})\geq j\right]\}\rho_{i\ell} = \sum_{j=1}^{M} b_j P\left[\phi(0_i,\underline{X})\geq j\right]$ $+ \sum_{\ell=1}^{M} I_\ell(i)\rho_{i\ell} = \sum_{j=1}^{M} b_j P\left[\phi(0_i,\underline{X})\geq j\right] + \underline{I}(i)\cdot\underline{\rho}_i^T$.

From this proposition we immediately have the following by partial differentiation.

<u>Proposition 4.2</u>. $\underline{I}(i) = \text{grad } U(\underline{\rho})$ where ρ_{jk} are treated as fixed for $j \neq i$ and $k = 1,\ldots, M$. This proposition says that if the marginals for X_j, $j \neq i$ are fixed and expected utility is viewed as a function of the i^{th} component, $\underline{I}(i) = \text{grad } U(\underline{\rho})$.

<u>Proposition 4.3</u>. If $a_j = j$ so that $U(\underline{\rho}) = E_{\underline{\rho}}\left[\phi(\underline{X})\right]$ then

$$\underline{I}(i)=\Big(\sum_{j=1}^{M}\{P\left[\phi(1_i,\underline{X})\geq j\right]-P\left[\phi(0_i,\underline{X})\geq j\right]\},\ \sum_{j=1}^{M}\{P\left[\phi(2_i,\underline{X})\geq j\right]-P\left[\phi(1_i,\underline{X})\geq j\right]\}$$
$$,\ldots,\ \sum_{j=1}^{M}\{P\left[\phi(M_i,\underline{X})\geq j\right]-P\left[\phi((M-1)_i,\underline{X})\geq j\right]\}\Big)=\Big(E\left[\phi(1_i,\underline{X})\right]-E\left[\phi(0_i,\underline{X})\right],$$
$$E\left[\phi(2_i,\underline{X})\right]-E\left[\phi(1_i,\underline{X})\right],\ldots,\ E\left[\phi(M_i,\underline{X})\right]-E\left[\phi((M-1)_i,\underline{X})\right]\Big).$$

<u>Proof</u>: If $a_j = j$ for $j = 0,1,\ldots, M$, then $b_j = 1$ for $j = 1,2,\ldots, M$. The conclusions of this proposition follow from the arguments given in the proof of Proposition 4.1.

Proposition 4.4. If component i is stochastically improved from a distribution $\underline{\rho}_i$ to a distribution $\underline{\rho}_i^* \geq \underline{\rho}_i$ then the change in the expected utility is $\underline{I}(i) \cdot \underline{\Delta}^T$ where $\Delta_j = \rho_{ij}^* - \rho_{ij}$ for all $j = 1,2,\ldots, M$.

Proof: Immediate from Proposition 4.1.

In the case of binary systems of binary components, it is possible to order the components by Birnbaum's reliability importance if one is interested in determining the influence of components on system reliability. In the multistate case, for any given improvement vector $\underline{\Delta}$, one can calculate $\underline{I}(i) \cdot \underline{\Delta}^T$ and $\underline{I}(j) \cdot \underline{\Delta}^T$. In general, however, the rankings depend on $\underline{\Delta}$ unlike the binary case where the rankings were independent of the improvement Δ. In a certain interesting case, however, we can obtain a ranking which does not depend on $\underline{\Delta}$.

Proposition 4.5. If the improvement vector is $\underline{\Delta} = \Delta\underline{1}$ where Δ is a positive scalar, then the expected utility is increased more by applying the improvement vector to component i than by applying it to component j if and only if $||\underline{I}(i)|| \geq ||\underline{I}(j)||$ where $||\underline{V}|| = \sum_{i=1}^{M} |V_i|$.

Proof: If $\underline{\Delta} = (\Delta,\Delta,\ldots, \Delta)$, then the improvement in expected utility due to an improvement of $\underline{\Delta}$ in component i is $\Delta||\underline{I}(i)||$. Likewise the corresponding improvement in expected utility if the improvement is applied to component j is $\Delta||\underline{I}(j)||$. The result then follows immediately.

The situation described in the hypothesis occurs, for instance, in the case that the components are actually binary so that $P[X_i=1] = \ldots = P[X_i=M-1] = 0$ for all i. Then improving component i would mean increasing $P[X_i=M]$ by Δ thus decreasing $P[X_i=0]$ by Δ, thereby obtaining an improvement vector of $\Delta\underline{1}$.

Finally, we consider an example which illustrates the use of this last proposition.

Example 4.1. Consider an n component system in which $P[X_i=0] = p_{10}$, $P[X_i=n] = p_{in}$ with $p_{10} + p_{in} = 1$ for all i. Then $P[X_i=j] = 0$ for $1 \leq j \leq n-1$ and all i. Consider the structure function $\phi(\underline{x}) = \left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]$ which is stochastically equal to the number of components in state n. Further assume that the utilities attached to the various states are $0 = a_0 \leq a_1 \leq \ldots \leq a_n$ and let $b_k = a_k - a_{k-1}$ for $1 \leq k \leq n$. For sake of definiteness we will compute $\underline{I}(1)$ and $\underline{I}(2)$. We will compare these under certain assumptions on the utility function.

By definition, $I_\ell(1) = \sum_{j=1}^{n} b_j I_{\ell j}(1)$, where $I_{\ell j}(1) = P[\phi(\ell, X_2, \ldots, X_n) \geq j] - P[\phi(\ell-1, X_2, \ldots, X_n) \geq j]$.

For $\ell < n$, $I_{\ell j}(1) = 0$,

$$\text{and } I_{nj}(1) = P\left[\sum_{i=2}^{n} X_i = (j-1)n\right].$$

Hence for $\ell < n$, $I_\ell(1) = \sum_{j=1}^{n} b_j I_{\ell j}(1) = 0$

$$\text{and } I_n(1) = \sum_{j=1}^{n} b_j I_{nj}(1) = \sum_{j=1}^{n} b_j P\left[\sum_{i=2}^{n} X_i = (j-1)n\right].$$

$$\text{Consequently } ||I(1)|| = \sum_{j=1}^{n} b_j \, P\left[\sum_{i=2}^{n} X_i = (j-1)n\right]$$

$$= \sum_{j=1}^{n-1} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-1)n,\ X_2 = 0\right]$$

$$+ \sum_{j=2}^{n} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-2)n,\ X_2 = n\right]$$

$$= \sum_{j=1}^{n-1} b_j (1-p_{2n}) \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right]$$

$$+ \sum_{j=2}^{n} b_j \, p_{2n} \, P\left[\sum_{i=3}^{n} X_i = (j-2)n\right]$$

$$= \sum_{j=1}^{n-1} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right]$$

$$- p_{2n} \sum_{j=1}^{n-1} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right]$$

$$+ p_{2n} \sum_{j=2}^{n} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-2)n\right]$$

$$= \sum_{j=1}^{n-1} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right]$$

$$- p_{2n} \sum_{j=1}^{n-1} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right]$$

$$+ p_{2n} \sum_{k=1}^{n-1} b_{k+1} \, P\left[\sum_{i=3}^{n} X_i = (k-1)n\right]$$

$$= \sum_{j=1}^{n-1} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right] + p_{2n} \sum_{j=1}^{n-1} (b_{j+1} - b_j) \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right]$$

In a completely analogous fashion, it can be shown that

$$||I(2)|| = \sum_{j=1}^{n-1} b_j \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right] + p_{1n} \sum_{j=1}^{n-1} \left\{(b_{j+1} - b_j) \, P\left[\sum_{i=3}^{n} X_i = (j-1)n\right]\right\}.$$

Next we consider three types of utility functions for which a comparison between $||\underline{I}(1)||$ and $||\underline{I}(2)||$ may be readily made. If the utility function is linear then $||\underline{I}(1)|| = ||\underline{I}(2)||$. If the utility function is convex, so that the b_i's are increasing, then $||\underline{I}(1)|| \geq ||\underline{I}(2)||$ if and only if $p_{2n} \geq p_{1n}$. In other words, the more important component is the less reliable one if the utility function is convex. In the case where the utility function is concave the situation is reversed. In this case $||\underline{I}(1)|| \geq ||\underline{I}(2)||$ if and only if $p_{2n} \leq p_{1n}$. That is, for a concave utility function the more important component is the more reliable one.

Thus for increasing the system's expected utility the components are ordered in importance (from most to least) by their reliabilities if the utility function is concave and by their unreliabilities if the utility function is convex.

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